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s-NUMBERS, MEASURE AND WEAK MEASURE OF NONCOMPACTNESS AND REAL INTERPOLATION

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The basic methods of interpolation theory were constructed to give quantitative results about norms of operators. It was, however, soon discovered that qualitative properties like compactness, degree of compactness, and weak compactness could also be interpolated. In this survey, we have collected and ordered some of this (partly very new) knowledge.

1. Introduction

The theory of interpolation spaces has its origin in the classical theorem of Riesz-Thorin. The Riesz-Thorin theorem states that if $p_0, p_1, q_0, q_1 \in [1, \infty]$ and $p_0 \neq p_1$, $q_0 \neq q_1$ and that $T : L_{p_0} \to L_{q_0}$ with norm M_0 and $T : L_{p1} \to L_{q1}$ with norm M_1, then $T : L_p \to L_q$ with norm M such that

$$M \leq M_0^{1-\theta} . M_1^{\theta} \tag{*}$$

provided $\theta \in (0, 1)$, i.e. log M is convex. An application of this theorem to Fourier transforms yields the Hausdorff-Young Inequality. In complex analysis, the conclusion of the Hadamard three circle theorem is like equation (*). On the other hand, in approximation theory, as we show later, Bernstein or Jackson type inequalities can be rewritten as convexity inequalities (*) of Riesz-Thorin. In the following we consider s-numbers for "degree of compactness", measure of non-compactness and weak measure of non-compactness and look at the convexity inequalities of type (*) for these concepts.

Some results about interpolation of eigenvalues are presented due to the close

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relationships between eigenvalues of compact operators and s-numbers and representation of radius of the essential spectrum in terms of measures of non-compactness.

2. s-numbers, Entropy Numbers

The concept of s numbers was invented by E. Schmidth in 1907, where he established a representation for integral operators induced by arbitrary continuous kernels. He referred to these quantities as eigenvalues. Later, in 1937 F. Smithies used the term *singular value*. For a long time, the understanding was:

The sequence of s numbers		The sequence of the
of an operator acting on a	:=	eigenvalues of the
Hilbert space		positive operator $\mid T \mid = (TT^*)^{1/2}$

After the formulation of D.E. Allakhverdiev in 1957 as

$$s_n(T) = \inf \{ \| T - L \| : L \in \mathcal{L}(H), \text{rank}(L) < n \}$$

it became possible to extend this notion to operators acting on Banach spaces. According to Pietsch [25], a map s which to each bounded linear map T from one Banach space to another such space assigns a unique sequence $(s_n(T))$ is called an *s-function* if for all Banach spaces E, F, G, W the following conditions are satisfied:

(i) $\| T \| = s_1(T) \geq s_2(T) \geq \ldots \geq 0$, for all $T \in \mathcal{L}(E, F)$.

(ii) $s_n(S+T) \leq s_n(S) + \| T \|$, for $S, T \in \mathcal{L}(E, F)$ and all $n \in N$.

(iii) $s_n(RST) \leq \| R \| \, s_n(S) \, \| T \|$, for $T \in \mathcal{L}(E, F)$, $S \in \mathcal{L}(F, G)$, $R \in \mathcal{L}(G, W)$ and $n \in N$.

(iv) If $T \in \mathcal{L}(E, F)$ and rank $(T) < n \in N$, then $s_n(T) = 0$.

(v) $s_n(I) = 1$ for all $n \in N$, where I is the identity map of $l_n^2 = \{x \in l^2 : x_i = 0$ if $i > n\}$ to itself.

$s_n(T)$ is called the n-th s-number of the operator T. Moreover, an s-function is called additive if

$$s_{n+m-1}(S+T) \leq s_n(S) + s_m(T), \; S, T \in \mathcal{L}(E, F).$$

Similarly, an s-function is called multiplicative, if

$$s_{n+m-1}(ST) \leq s_n(S)\, s_m(T), \; T \in \mathcal{L}(E, F), S \in \mathcal{L}(F, G).$$

Now we turn to some special s-numbers. Their definition are:

-Approximation numbers :

$$a_n(T) := \inf\{\| T - S \| : \operatorname{rank}(S) \le n\},\ T, S \in \mathcal{L}(E, F).$$

-Gelfand numbers:

$$c_n(T) := \inf\{\| T J_M^E \| \ \operatorname{Codim}(M) < n\},\ T \in \mathcal{L}(E, F),$$

where J_M^E is the embedding map from M into E.

-Kolmogorov numbers (or n-widths):

$$d_n(T) := \inf\{\| Q_N^F T \| : \dim(N) < n\}, T \in \mathcal{L}(E, F),$$

where Q_N^E is the canonical map from F to F/N.

Observe that Kolmogorov numbers $d_n(T)$, $n = 1, 2, \ldots$ can also be defined as

$$d_n(T) := \inf_{N \subset F} \sup_{x \in B_E} \inf_{y \in N} \| Tx - y \|,$$

where N is an arbitrary subspace of F with $\dim N < n$. The above definition of Kolmogorov numbers illustrates the fact that how "good" the image $T(B_E)$ can be approximated by $(n-1)$-dimensional subspaces of F.

In Hilbert spaces, all s-number sequences coincide. In Banach space, the approximation numbers are the largest. For relations between several kinds of s-numbers we refer to **[26]**. The s-numbers listed above are both additive and multiplicative; s-numbers characterize the "degree of compactness" as well as the radius of the essential spectrum **[17]**.

Let $T \in \mathcal{L}(E, F)$ and let $n \in N$. The n-th entropy number $e_n(T)$ of T is defined by

$$e_n(T) := \inf\{\varepsilon > 0 : T(B_E) \subset \bigcup_{i=1}^{2^{n-1}} (y_i + \varepsilon B_F) \text{ for some } y_i \in F,\ i = 1, 2, \ldots, 2^{n-1}\}.$$

The set B_E and B_F denote the closed unit balls of E and F respectively.

Since $\| T \| = e_1(T) \ge e_2(T) \ge \ldots \ge 0$, for $T \in \mathcal{L}(E, F)$ the sequence $(e_n(T))$ is monotone decreasing as n increases, and so the limit exists. Clearly,

$$\lim_{n\to\infty} e_n(T) = \inf\{\varepsilon > 0 : T(B_E) \text{ can be covered by finitely many balls of radius } \varepsilon\}$$

Therefore $\lim_{n\to\infty} e_n(T) = \tilde{\beta}(T)$ where $\tilde{\beta}(T) = \tilde{\beta}(T(B_E)) := \inf\{\varepsilon > 0 : T(B_E) \subset \cup_{i=1}^{k} B(y_i; \varepsilon)\}$ is the ball measure of non-compactness of T; $B(y_i; \varepsilon)$ stands for the ball center at $y_i \in F$ and radius ε. In [2], it is proved that $\lim_{n\to\infty} d_n(T) = \tilde{\beta}(T)$. We also know $\frac{1}{2}\tilde{\beta}(T) \leq c(T) \leq 2\tilde{\beta}(T)$ holds [10].

The following gives an estimate between s-numbers and the dyadic entropy numbers.

THEOREM 1 ([6]) *Let* $p \in (0, \infty)$ *and* $s \in \{a, c, d\}$. *Then for all Banach spaces E and F and all* $T \in \mathcal{L}(E, F)$, *the inequlatiy*

$$\sup_{1\leq k\leq n} k^{1/p} e_k(T) \leq \rho_p \sup_{1\leq k\leq n} k^{1/p} s_k(T), \text{ for } n = 1, 2, \ldots$$

is valid.

3. Eigenvalues

Recently a great deal of work has been done to relate the analytical entities related to bounded linear maps such as eigenvalues, essential spectrum, and the geometrical quantities such as entropy numbers, approximation numbers and n-widths. Although these connections are interesting in themselves, there is a definite use of this theory in the theory of partial differential equations.

Also using various s-numbers or entropy numbers one can describe "degree of compactness" for an operator. But one would like to describe certain characteristics for the "degree of compactness" which imply a good approximation as well as good behaviour for its eigenvalues. In this section we only sketch the most important results in this direction. Those readers interested in the full theory can consult Peitsch [25] or Konig [13].

Throughout this paper E, F will denote the complex Banach space. Suppose $T \in \mathcal{L}(E)$ is compact. Compact operators have strikingly nice properties. In particular, the spectrum of T, apart from the point 0, consists solely of eigenvalues of finite algebraic multiplicity. $n(\lambda; T) := \dim\{x \in E : (\lambda I - T)^k = 0 \text{ for some } k\}$. Also the set of eigenvalues does not possess any accumulation point different from zero. Consequently, for every compact operator $T \in \mathcal{L}(E)$, one can associate its ***eigenvalue sequence*** $(\lambda_n(T))$ which is defined as follows :

(i) Every eigenvalue $\lambda \neq 0$ is counted according to its multiplicity i.e. it occurs

$n(\lambda; T)$ times, one after the other.

(ii) The eigenvalues are arranged in order of non-increasing magnitude, i.e.,

$|\lambda_1(T)| \geq |\lambda_2(T)| \geq \ldots \geq 0$.

(iii) If T has less than n eigenvalues $\lambda \neq 0$ then set $\lambda_n(T) := 0$.

We start with the following remarkable formula due to Konig [14].

THEOREM 2 *Let $T \in \mathcal{L}(E)$ be compact. Then for all $n \in N$,*

$$|\lambda_n(T)| = \lim_{k \to \infty} [a_n(T^k)]^{1/k}$$

Notice that for $n = 1$, this is the spectral radius formula since $a_1(T) = \|T\|$. The proof uses induction on n to establish

$$\lim \sup_{k \to \infty} [a_n(T^k)]^{1/k} \leq |\lambda_n(T)| \leq \lim \inf_{k \to \infty} [a_n(T^k)]^{1/k}$$

details can be found in [14].

If H is a complex *Hilbert space* for $T \in \mathcal{L}(H)$ compact we set $|T| = \mathcal{U}|T|$ where $\mathcal{U}$ is a partial isometry. So if *T is compact so is $|T|$, by the ideal property. it is known that for the Hilbert space case with $|T|$ compact on E*, for all $n \in N$ we have $a_n(T) = \lambda_n(|T|)$, where $(\lambda_n(|T|))$ is the eigenvalue sequence for $|T|$. But even in this Hilbert space case no inequality of the form $a_n(T) \geq (\leq) |\lambda_n(T)|$ for all $n \in N$ can be expected. For example, $E = C^2$ det $T \in \mathcal{L}(C^2)$ can be represented by $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ then $\lambda_1(T) = 2$, $\lambda_2(T) = 1$. But $|T|^2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$ has eigenvalues $= 3 \pm \sqrt{5}$ so $a_1(T) = \lambda_1(|T|) = \sqrt{3+\sqrt{5}} > 2 = |\lambda_1(T)|$ and $a_2(T) = \lambda_2(|T|) = \sqrt{3-\sqrt{5}} < 1 = |\lambda_2(T)|$.

In contrast to approximation numbers, there is an estimate for a single eigenvalue $\lambda_n(T)$ in terms of a single entropy number $e_n(T)$. The following theorem is due to Carl [6].

THEOREM 3 *Let $T \in \mathcal{L}(E)$ be a compact operator and let $(\lambda_n(T))$ be an eigenvalue sequence. Then for all $m, n \in N$*

$$|\lambda_n(T)| \leq \prod_{j=1}^{n} (|\lambda_j(T)|)^{1/n} \leq (\sqrt{2})^{m-1/n} e_m(T).$$

Note that by taking $m = n + 1$, the above theorem gives the inequality

$$|\lambda_n(T)| \le \sqrt{2}\, e_{n+1}(T).$$

This theorem also gives rise to the following corollary due to Zemanek [**30**].

COROLLARY 1 *Let* $T \in \mathcal{K}(X)$. *Then, for all* $n \in N$,

$$r(T) = \lim_{k\to\infty} (e_n(T^k))^{1/k}$$

the spectrial radius of T.

It should be mentioned that for a compact operator acting on a Hilbert space H, one has the following well known Weyl inquality:

$$\Sigma_{k=1}^{\infty} |\lambda_k(T)|^p \le \Sigma_{k=1}^{\infty} a_k(T)^p$$

which shows that if the approximation numbers from a sequence in l^p with $0 < p < \infty$, then so do the eigenvalues. Here are some other versions of the Weyl inequality:

additive Weyl inequality $\qquad \Sigma_{k=1}^{n} |\lambda_k(T)|^p \le \Sigma_{k=1}^{n} a_k(T)^p$

for $n = 1, 2, \ldots$

multiplicative Weyl inequality $\qquad \Pi_{k=1}^{n} |\lambda_k(T)|^p \le \Pi_{k=1}^{n} a_k(T)^p$

for $n = 1, 2, \ldots$.

Also, inequalities of the type

$$\left(\Sigma_{k=1}^{\infty} |\lambda_k(T)|^p\right)^{1/p} \le K_p \left(\Sigma_{k=1}^{\infty} a_k(T)^p\right)^{1/p}$$

exist in Banach space settings. Here $T \in \mathcal{L}(E)$ must be compact, $p \in (0, \infty)$ and the constant K_p depends only on p.

Again in the setting of a general Banach space, using Lorentz sequence spaces $l_{\frac{1}{1-\theta}, q} = (l_1, l_\infty)_{\theta, q}$, Edmunds gives [10], a simple proof of the Weyl inequality. He shows that if the n-th approximation number is $O(n^{-\alpha})$ for some $\alpha > 0$, as $n \to \infty$, then so is the n-th eigenvalue.

Let $\mathcal{K}$ be the closed ideal of compact operators on a Banach space E. The quotient algebra $\mathcal{L}(E)/\mathcal{K}$ is a Banach algebra called the Calkin algebra. The essential spectrum $\sigma_e(T)$ of T is defined by:

$$\sigma_e(T) = \{\lambda \in C : \pi(\lambda - T) \text{ is not invertible in } \mathcal{L}(E)/\mathcal{K}\}$$

where π is the natural homomorphism from $\mathcal{L}(E)$ one $\mathcal{L}(E)/\mathcal{K}$. Nussbaum [22] uses the concept of ball measurement of noncompactness to obtain a formula for $r_e(T)$, the radius of the essential spectrum. Namely, he shows that

$$r_e(T) = \lim_{n\to\infty} (\tilde{\beta}(T^n))^{1/n}$$

where as before $\tilde{\beta}(T)$ denotes the ball measure of non-compactness of T. Several papers were written, such as [16], [9], to show that:

$$r_e(T) = \lim_{n\to\infty} (s(T^n))^{1/n}$$

where $s \in \{c, d\}$ and $c(T) = \lim_n c_n(T)$, $d(T) = \lim_n d_n(T)$. Also, in [1] one finds the minimum conditions on the s-numbers in order for the above equation to hold.

4. Interpolation Spaces

Let $\tilde{E} = \{E_0, E_1\}$ be a *Banach couple*, i.e. E_0 and E_1 are Banach spaces continuously imbedded into a Hausdorff topological vector space. For a Banach couple $\tilde{E} = \{E_0, E_1\}$ we can form the *intersection* $\Delta(\tilde{E}) = E_0 \cap E_1$ and the sum $\Sigma(\tilde{E}) = E_0 + E_1$. They are both Banach spaces with the norms :

$$\| a \|_{\Delta(\tilde{E})} := \max\{\| a \|_{E_0}, \| a \|_{E_1}\}$$

and

$$\| a \|_{\Sigma(\tilde{E})} := K(t, a; \tilde{E})$$

respectively. Here for $t > 0$

$$K(t, a; \tilde{E}) = \inf\{\| a_0 \|_{E_0} + t \| a_1 \|_{E_1} : a = a_0 + a_1,\ a_0 \in E_0,\ a_1 \in E_1\}$$

is the *K-functional* of Peetre.

A Banach space E is called an *intermediate space* between E_0 and E_1 (or with respect to $\tilde{E}$) if

$$\Delta(\tilde{E}) \subset E \subset \Sigma(\tilde{A}),$$

and the corresponding embeddings are continuous. If in addition every bounded operator in $\Sigma(\tilde{E})$ that leave E_0 and E_1 invariant also maps E boundedly into itself, then E is called an *interpolation space* between E_0 and E_1 (or with respect to $\tilde{E}$).

Let $\mathcal{L}(\{E_0, E_1\}, \{F_0, F_1\})$ be the Banach space of all operators $T: E_0 + E_1 \to F_0 + F_1$ such that the restriction of T to the space E_i is a bounded operator from E_i into F_i, $i = 0, 1$, with norms

$$\| T \|_{\mathcal{L}(\{E_0, E_1\}, \{F_0, F_1\})} = \max \{ \| T_{E_0, F_0} \|, \| T_{E_1, F_1} \| \} .$$

We say two intermediate spaces E and F are *interpolation spaces of exponent* θ $(0 < \theta < 1)$ if given any $T \in \mathcal{L}(\{E_0, E_1\}, \{F_0, F_1\})$ the restriction of T to E is in $\mathcal{L}(E, F)$ and

$$\| T_{E, F} \| \leq \| T_{E_0, F_0} \|^{1-\theta} \| T_{E_1, F_1} \|^{\theta}$$

There are several methods [20] of constructing interpolation spaces of exponent θ with respect to Banach pairs $\tilde{E}$ and $\tilde{F}$ The real method is defined as follows : for $0 < \theta < 1$ and $1 \leq p \leq \infty$, if $p < \infty$:

$$\tilde{E}_{\theta,p} = (E_0, E_1)_{\theta,p} = \{ a \in E_0 + E_1 : \| a \|_{\theta,p} = \left(\int_0^\infty [t^{-\theta} K(t; a)]^p \frac{dt}{t} \right)^{1/p} < \infty \}$$

if $p = \infty$ then

$$E_{\theta,\infty} = (E_0, E_1)_{\theta,\infty} = \{ a \in E_0 + E_1 : \| a \|_{\theta,\infty} = \sup_{0<t<\infty} t^{-\theta} K(t, a) < \infty \} .$$

It can be shown that if $a \in E_0 \cap E_1$, then

$$\| a \|_{\theta,p} \leq \| a \|_{E_0}^{1-\theta} \| a \|_{E_1}^{\theta} .$$

Here are some classical examples of interpolation spaces [21] :

1) $(L_1, L_\infty)_{\theta,q} = L_{p,q}$ (Lorentz space) if $\frac{1}{p} = 1 - \theta$, then $0 < \theta < 1$.

Note that in particular

$$L_{p,p} = L_p \text{ (Lebesque space)}$$

$$L_{p,\infty} = L_p^* \text{ (Weak Lebesque or Marcinkiewicz space)}$$

2) if $p_0, p_1 \in [1, \infty]$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ with $0 < \theta < 1$, then

$$(W^{k,p_0}(\Omega), W^{k,p_1}(\Omega))_{\theta,p} = W^{k,p}(\Omega)$$

where $W^{k,p}(\Omega)$, for $p \in [1, \infty]$, $k = 0, 1, 2, ..., \Omega \subset R^n$, denote the Sobolev space of all $f \in L^p(\Omega)$ which have $D^\alpha f$ for $0 \le |\alpha| \le k$ and for which

$$\|f\|_{k,p} = \Sigma_{|\alpha| \le k} \|D^\alpha f\|_p < \infty .$$

In the following discussion we shall refer to the restriction of T to $E = (E_0, E_1)_{\theta,p}$, viewed as an element of $\mathcal{L}(E, F)$ where $F = (F_0, F_1)_{\theta,p}$ as $T_{E,F}$; T_{E_j, F_j}, $j = 0, 1$ will have a similar meaning.

Finally, given E and F intermediate spaces with respect to $\tilde{E}, \tilde{F}$ we say that E possesses *K-type* θ, $\theta \in (0, 1)$, if there is a positive constant c such that for all $a \in E$ and all $t > 0$

$$K(t; a) \le ct^\theta \|a\|_E \quad \text{for all } a \in E .$$

F possesses *J-type* θ, there is a positive constant c such that

$$\|b\|_F \le c \|b\|_{F_0}^{1-\theta} \|b\|_{F_1}^{\theta} \quad \text{for all } b \in F_0 \cap F_1 .$$

Real interpolation spaces possesses both *K-type* θ and *J-type* θ. We also have:

THEOREM 4 [25] *Let (Ω, μ) be any Σ- finite measure space. Furthermore suppose that $p_0, p_1 \in [1, \infty]$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then $L_p(\Omega, \mu)$ is an intermediate space of $\{L_{p_0}(\Omega, \mu), L_{p_1}(\Omega, \mu)\}$. Moreover, $L_p(\Omega, \mu)$ has K-type θ and J- type θ.*

In 1960, Krasnosel'skii [16] proved the following version of the Riesz-Thorin theorem for compact operators: let $T : L_{p_0} \to L_{p_0}$ be bounded and $T : L_{p_1} \to L_{q_1}$ be compact, where all four exponents are in the range $[1, \infty]$ and $q_0 < \infty$. Then $T : L_{p_\theta} \to L_{q_\theta}$ is compact too. Here $0 < \theta < 1$ and

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \frac{1}{q_0} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} .$$

In light of the Krasnosel'skii theorem it is natural to ask, given two Banach couples $\tilde{E}$ and $\tilde{F}$ and interpolation spaces E and F obtained by the real method with respect to $\tilde{E}$ and $\tilde{F}$ whether T viewed as a map from E to F inherits any compactness properties which it may possess as an element of $\mathcal{L}(E_i, F_i)$?

One sided answers were given by Lions and Peetre [18]. They showed that if $F_0 = F_1$ and E is of *K-type* θ for some $\theta \in (0, 1)$, then $T : E \to F_0$ is compact if either $T : E_0 \to F_0$ or $T : E_1 \to F_0$ is compact. They also have similar results when

$F_0 \neq F_1$ and $E_0 = E_1$.

The general case, in which $E_0 \neq E_1$ and $F_0 \neq F_1$ was solved by Persson [24]. He shows that if $T: E_0 \to F_0$ is compact so is $T: E \to F$. To do this he has forced to make the assumption the $\{F_0, F_1\}$ has certain approximation properties (H). The best result in this direction was recently proved by Cwikel [8]. He shows that if $T: E_0 \to F_0$ is compact and $T: E_1 \to F_1$ is bounded, then $T: (E_0, E_1)_{\theta,p} \to (F_0, F_1)_{\theta,p}$ is compact too. For our discussion in the next section we need the following approximation hypothesis-(H) defined by Persson.

Approximation hypothesis-(H): We say that a Banach couple $\{F_0, F_1\}$ has the approximation hypothesis-(H), if there exists a positive constant c such that given any $\in > 0$ and any finite sets $B_0 \subset F_0$ and $B_1 \subset F_1$, there is an operator $P \in \mathcal{L}(F, \tilde{F})$ such that

1) $P(F_i) \subset F_1 \cap F_2$ for $i = 0, 1$,

2) $\| P \|_{\mathcal{L}(E_i, E_i)} \leq c \quad i = 0, 1$,

3) $\| x - Px \|_{E_i} < \varepsilon$ for all $x \in B_i$, $i = 0, 1$,

Following is an example of an interpolation pair satisfying the approximation hypothesis-(H).

PROPOSITION 1 [24] *Let X be a locally compact space with positive measure μ let $p_0, p_1 \in [1, \infty)$. Then $\{L_{p_0}(X, \mu), L_{p_1}(X, \mu)\}$ is an interpolation pair which satisfies the approximation hypothesis -(H).*

Proof of the above proposition can be found in [24], but the idea is to consider $B_0 \subset L_{p_0}$, $B_1 \subset Lp_1$ finite sets and to let S be the set of all bounded measurable functions with compact support, one may assume $B_0, B_1 \subset S$. Let $K \subset X$ be a compact set in X outside which all $f \in B_i$, $i = 0, 1$ vanish. Choose $\eta > 0$ such that $\eta \cdot \max(1, \mu(K)) < \varepsilon$. Construct a finite partition (K_n) of K consisting of a set $\mu(K_0) = 0$ and measurable sets $K_1, K_2, \ldots, K_N$, with $\mu(K_n) > 0$ and $\sup_{x,y \in K_j} | f(x) - f(y) | < \eta$, $j = 1, 2, \ldots, N$, and, for all $f \in B_i$, define

$$P_f = \Sigma_{n>0} \left(\frac{\int_{K_n} f \, d\mu}{\mu(K_n)} \right) \chi K_n$$

for locally integrable functions f. Then it is obvious that $P(L_{p_i}) \subset L_{p_0} \cap L_{p_i}$, $i = 0, 1$ and moreover $\| Pf \|_{L_p}^p \leq \| f \|_{L_p}^p$, $\| Pf - f \|_{L_{p_i}} < \varepsilon$ for all $f \in B_i$.

5. Real Interpolation, s-numbers, Measure and Weak Measure of Noncompactness

In view of all this work on *compact* operators, two natural concerns arise: first, one may like to investigate the behavior under interpolation of *weakly compact* operators, secondly one might like to enquire into the behaviour under interpolation of properties, which may mean more than mere continuity yet not so much as compactness. The *s-numbers* and *measures of non-compactness* come immediately to mind.

In this section we focus on these problems. We start with the following theorem of Pietsch.

THEOREM 5 [25]

a) *Let E be an intermediate space of $\{E_0, E_1\}$ possessing K-type θ. If $T \in \mathcal{L}(\Sigma(\widetilde{E}), F)$, then*

$$d_{n_0+n_1-1}(T_{E,F}) \le d_{n_0}(T_{E_0,F})^{1-\theta}\, d_{n_1}(T_{E_1,F})^{\theta},$$

$$e_{n_0+n_1-1}(T_{E,F}) \le 2e_{n_0}(T_{E_0,F})^{1-\theta}\, e_{n_1}(T_{E_1,F})^{\theta}.$$

b) *Let F be an intermediate space of $\{F_0, F_1\}$ possessing J-type θ. If $T \in \mathcal{L}(E, \Delta(\widetilde{F}))$, then*

$$c_{n_0+n_1-1}(T_{E,F}) \le c_{n_0}(T_{E,F_0})^{1-\theta}\, c_{n_1}(T_{E,F_1})^{\theta},$$

$$e_{n_0+n_1-1}(T_{E,F}) \le 2e_{n_0}(T_{E,F_0})^{1-\theta}\, e_{n_1}(T_{E,F_1})^{\theta}$$

where d_n, c_n, and e_n are the Kolmogorov numbers, Gelfand numbers and entropy numbers respectively.

In order to obtain interpolation theorems for the s-finctions ($s \in \{c, d\}$) in the general case, where $E_0 \ne E_1$ and $F_0 \ne F_1$, Teixeira [27] considers approximation property-(H) on the space $\{F_0, F_1\}$ and proves an inequality of the type

$$d_{n_0+n_1-1}(T_{E,F}) \le c\,(1+\sqrt{n_0+n_1-1})\, d_{n_0}(T_{E_0,F_0})^{1-\theta}\, d_{n_1}(T_{E_1,F_1})^{\theta}.$$

A corresponding inequality is proved for Gelfand numbers.

Although his estimates are weaker than those established in the above proposition, using the Konig-Zemanek result [14], [30] and the fact that $d_{2n} \leq d_{2n-1}$ for all $n \in N$, he obtains the following.

PROPOSITION 2 [27] *Let* $\{E_0, E_1\}$ *be an interpolation pair of Banach spaces and let* $E = (E_0, E_1)_{\theta,p}$ *for some* $\theta \in (0, 1)$ *and some* $p \in [1, \infty]$. *Suppose* $\{E_0, E_1\}$ *has the approximation property-(H). Then if* $T \in \mathcal{L}(\vec{E}, \vec{E})$, *we have*

$$|\lambda_{2n-1}(T_{E,E})| \leq |\lambda_n(T_{E_0,E_0})|^{1-\theta} |\lambda_n(T_{E_1,E_1})|^{\theta}$$

for all $n \in N$.

The study of interpolation of entropy numbers for the general case when $E_0 \neq E_1$ and $F_0 \neq F_1$ was also done by M.F. Teixeira [28]. However, to do this he was forced to make the assumptions that:

1) $\{F_0, F_1\}$ has the approximation property-(H) and

2) the map $T_{E_0,F_0} : E_0 \to F_0$ is compact.

Under these two conditions, he shows:

$$e_{n_0+n_1-1}(T_{E,F}) \leq 2c e_{n_0}(T_{E_0,F_0})^{1-\theta} e_{n_1}(T_{E_1,F_1})^{\theta}.$$

this inequality is used to obtain interpolation of the so-called entropy ideals. A comprehensive account of interpolation of entropy ideals and width ideals is given in [29]. It should be mentioned that quasi-Banach ideals associated with the approximation numbers behave better with respect to interpolation than those corresponding to other s-numbers. To illustrate this, we need:

DEFINITION. Let $l_{p,q}$ be the Lorentz sequence space. Then the Lorentz operator ideal denoted by $\mathcal{L}^{(s)}_{p,q}(E, F)$ is defined as

$$\mathcal{L}^{(s)}_{p,q}(E, F) = \{T \in \mathcal{L}(E, F) : (s_n(T)) \in l_{p,q}\}$$

$$= \{T \in \mathcal{L}(E, F) : \left(\Sigma_{n=1}^{\infty} [n^{1/p-1/q} s_n(T)]^q\right)^{1/q} < \infty\}.$$

Konig [15] shows that if $p_0, p_1 \in (0, \infty)$ and $q_0, q_1, q \in (0, \infty)$ and $0 < \theta < 1$, and if $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, then

$$(\mathcal{L}^{(s)}_{p_0,q_0}, \mathcal{L}^{(s)}_{p_1,q_1})_{\theta,p} \subseteq \mathcal{L}^{(s)}_{p,q}$$

here s refers to s-numbers. But if $s = a$ the approximation numbers, under the same hypothesis, one has:

$$(\mathcal{L}^{(s)}_{p_0,q_0}, \mathcal{L}^{(s)}_{p_1,q_1})_{\theta,p} = \mathcal{L}^{(s)}_{p,q}$$

Proof of the above facts can be found in [26].

Next we look as interpolation of measure of non-compactness and measure of weak non-compactness. It turns out that natural analogues of the results of Lions and Peetre about one-sided compactness theorems can be carried out to measure or weak measure of non- compactness.

DEFINITION. Let E and F be Banach spaces, and let D be a bounded subset of E. The *measure of non- compactness* of D, $\alpha_E(D)$, is defined by

$$\alpha_E(D) = \inf\{\varepsilon > 0 : D \text{ can be covered by finitely many sets of diameter} \le \varepsilon\}$$

the ball measure of non-compactness defined in exactly the same way except that balls of radius $\le \varepsilon$ are used in place of arbitrary sets of diameter $\le \varepsilon$.

On the other hand, the *weak measure of non-compactness* of D, $w_E(D)$, is defined as:

$$w(D) = w_E(D) = \inf\{\varepsilon > 0 : D \subset \varepsilon B_E + W,\ W \subset E \text{ weakly compact}\}$$

where B_E denotes the closed unit ball of E. This concept was first introduced by De Blasi [9]. Now let $k \in R$, $k \ge 0$. A map $T \in \mathcal{L}(E, F)$ is called a *k-set contraction* if and only if

$$\alpha_F(T(D)) \le k\alpha_E(D) \text{ for all bounded sets } D \subset E,$$

and $\beta(T_{E,F}) = \beta(T) := \min\{k : T \text{ is a } k\text{-set contraction}\}$ is called the measure of non-compactness of T.

The circle of ideas around k-set contractions is now of proven importance in functional analysis and differential equations. The *weak k-set contraction* and weak measure of non-compactness of T, $w(T)$, defined analogously:

$$w(T) = w(T_{E,F}) := \min\{k : w_F(T(D)) \le k w_E(D) \text{ for all bounded sets } D \subset E\};$$

this concept has been applied to obtain fixed point theorems [12] and existence results for differential and functional equations in Banach spaces [4].

It is worth mentioning that

1) $\beta(T) = 0$ if and only if T is compact

$w(T)=0$ if and only if T is relatively weakly compact.

2) $w(T) \le \tilde{\beta}(T)$ where $\tilde{\beta}$ denotes the *ball* measure of non-compactness of T.

The following two theorems both have 2-parts: Part 1 in each was proved by Edmunds and Teixeira [11], part 2 in each are due to the author of this paper and L. Maligranda [3].

THEOREM 6 *Let $\{F_0, F_1\}$ be a Banach couple and E is a Banach space. Suppose $F=(F_0, F_1)_{\theta,p}$ possesses a J-type θ for some $\theta \in (0,1)$. If $T \in \mathcal{L}(\{E,E\}\{F_0, F_1\})$, then*

$$1)\ \beta(T_{E,F}) \le c\,(\beta(T_{E,F_0}))^{1-\theta}(\beta(T_{E,F_1}))^{\theta},$$

$$2)\ w(T_{E,F}) \le c\,(w(T_{E,F_0}))^{1-\theta}(w(T_{E,F_1}))^{\theta}.$$

Details of the proof are given in [3] and [11]. The main difference between the proofs of 1) and 2) is the fact that for 2) one needs to show that for a given Banach couple $\{F_0, F_1\}$, if W_0 and W_1 are weakly compact sets in the spaces F_0 and F_1 respectively, then $W_0 \cap W_1$ is weakly compact in $F_0 \cap F_1$.

THEOREM 7 *Let $\{E_0, E_1\}$ be a Banach couple and F be a Banach space. Suppose $E=(E_0, E_1)_{\theta,p}$ possesses a K-type θ; for some $\theta \in (0,1)$. If $T \in \mathcal{L}(\{E_0, E_1\}, \{F, F\})$, then*

$$1)\ \beta(T_{E,F}) \le c\,(1-\theta)^{\theta-1}\theta^{-\theta}(\beta(T_{E_0,F}))^{1-\theta}(\beta(T_{E_1,F}))^{\theta}$$

$$2)\ w(T_{E,F}) \le c\,(1-\theta)^{\theta-1}\theta^{-\theta}(w(T_{E_0,F}))^{1-\theta}(w(T_{E_1,F}))^{\theta}.$$

Concerning the general case where $E_0 \ne E_1$ and $F_0 \ne F_1$ requires a strengthening of Persson's approximation hypothesis-(H), under this assumption, in [11] it is proved that

$$\tilde{\beta}(T_{E,F}) \le c_0^{1-\theta} c_1^{\theta} (\tilde{\beta}(T_{E_0,F_0}))^{1-\theta} (\tilde{\beta}(T_{E_1,F_1}))^{\theta} \qquad (**)$$

is true.

One interesting consequence of such inequalities for a map $T \in \mathcal{L}(\{E_0, E_1\}, \{E_0, E_1\})$ is the following:

COROLLARY 2 *Let $\{E_0, E_1\}$ be an interpolation pair which satisfies the approximation hypothesis (H) and let $E=(E_0, E_1)_{\theta,p}$ for some $\theta \in (0,1)$ and some $p \in [1,\infty]$. If $T \in \mathcal{L}(\{E_0, E_1\}, \{E_0, E_1\})$ then for the radius of the essential spectrum*

r_e we have:

$$r_e(T_{E,E}) \le (r_e(T_{E_0,E_0}))^{1-\theta}(r_e(T_{E_1,E_1}))^{\theta}.$$

The proof follows immediately, by taking $E_0 = F_0$, $E_1 = F_1$ and T placed by T^n for any $n \in N$ in the above inequality (**) and using Nussbaum's formula $r_e(T) = \lim_{n\to\infty}(\tilde{\beta}(T^n))^{1/n}$

Interpolation of weakly compact operators were investigated by several authors. In [20], it is shown that, given $p \in (1,\infty)$ and $T \in \mathcal{L}(\{E_0, E_1\}, \{F_0, F_1\})$ and $E = (E_0, E_1)_{\theta,p}$, $F = (F_0, F_1)_{\theta,p}$ for some $\theta \in (0, 1)$, one has : $T: E \to F$ is weakly compact if and only if $T: E_0 \cap E_1 \to F_0 + F_1$ is weakly compact. Therefore it is natural to ask if one can improve inequalities about weak measure of non-compactness given in Theorems 6 and 7 above? The following is a partial answer to this question.

PROPOSITION 3 *Let* $\{E_0, E_1\}$ *be a Banach couple and* $E = (E_0, E_1)_{\theta,p}$. *If E is of K-type* θ, *then for every* $\varepsilon > 0$, *there exists* $t > 0$ *such that*

$$B_E \subset tB_{\Delta(\bar{E})} + \varepsilon B_{\Sigma(\bar{E})}$$

where B_E, $B_{\Delta(\bar{E})}$, *and* $B_{\Sigma(\bar{E})}$ *are unit balls of* E, $E_0 \cap E_1$, *and* $E_0 + E_1$ *respectively.*

PROOF. E is of K-*type* θ, $\theta \in (0, 1)$ if there exists a positive constant c such that for all $a \in E$ and all $t > 0$,

$$\inf\{\|a_0\|_{E_0} + t\|a_1\|_{E_1} : a = a_0 + a_1, a_0 \in E_0, a_1 \in E_1\} \le ct^{\theta}\|a\|_E.$$

Fix $\varepsilon > 0$. Then $\exists\, i, j \in N\; 2^{\theta i} < \frac{\varepsilon}{4}$ and $2^{\theta-1} j < \frac{\varepsilon}{4}c^{-1}$. If $x \in B_E$, then from the above we obtain

$$\|a_0\|_{E_0} + 2^{-i}\|a_1\|_{E_1} \le c2^{-\theta i} + \varepsilon/4$$

$$\|a'_0\|_{E_0} + 2^{j}\|a_1'\|_{E_1} \le c2^{\theta j} + \varepsilon/4$$

where $a = a_0 + a_1 = a_0' + a_1'$ are such that $a_0, a_0' \in E_0$ $a_1, a_1' \in E_1$. Next observe that $\|a_0\|_{E_0} < \varepsilon/2$. Let $b = a - a_0 - a_1' = a_0' - a_0 \in E_0$. Then $\|b\|_{E_0} \le \|a_0\|_{E_0} + \|a_0'\|_{E_0} < 2^j\varepsilon$. On the other hand $b = a_1 - a_1'$ therefore $\|b\|_{E_1} \le \|a_1\|_{E_1} + \|a_1'\|_{E_1} < 2^i\varepsilon$, which implies that $b \in E_0 \cap E_1$ and $\|b\|_{\Delta(\bar{E})} \le \max(2^i, 2^j)\,\varepsilon$. Now consider $a-b=a_0+a_1'$ and $\| a-b \|_{\Sigma(\tilde{E})} \leq \| a_0 \|_{E_0} + \| a_1' \|_{E_1} < \varepsilon$. Since $a=b+a_0+a_1'$ with $b \in \Delta(\tilde{E})$, $a_0+a_1' \in \Sigma(\tilde{E})$ we have

$$B_E \subset t^{-1} B_{\Delta(\tilde{E})} + \varepsilon^{-1} B_{\Sigma(\tilde{E})}$$

where $t = \max(2^i, 2^j)\varepsilon$.

The above properties imply the following improvement of Theorem 7, proof of which can be found in [3].

THEOREM 8 *Let* $\{E_0, E_1\}$ *be a Banach couple and* $E=(E_0, E_1)_{\theta,p}$ *for some* $\theta \in (0,1)$ *and* $p \in [1,\infty]$. *Suppose for every* $\varepsilon > 0$ *there exsits* $t > 0$ *such that*

$$B_E \subset tB_{\Delta(\tilde{E})} + \varepsilon B_{\Sigma(\tilde{E})}.$$

If $T \in \mathcal{L}(\{E_0, E_1\}, \{F, F\})$, *then*

$$w(T_{E,F}) \leq w(T_{E_0 \cap E_1, F}) \leq \frac{\left(\frac{1-\theta}{\theta}\right)^{\theta} + \left(\frac{\theta}{1-\theta}\right)^{1-\theta}}{4c} w(T_{E_0 \cap E_1, F})^{1-\theta} d^{\theta}$$

where $d = \max(\| T_{E_0,F} \|, \| T_{E_1,F} \|)$.

6. Bernstein-Jackson-Type Inequalities

6.1 Interpolation and Inequalities of Bernstein-Jackson-Type

When an inequality of the type

$$\| T_{E,F} \| \leq T_{E_0, F_0} \|^{1-\theta} \| T_{E_1, F_1} \|^{\theta}$$

appears, there is often a connection with interpolation theory. In the following it is shown that [5] classical inequalities of Bernstein and Jackson type in approximation theory can be reformulated as a convexity inequality above.

One may write the Bernstein inequality as follows:

$$\sup_T | D^j p(x) | \leq n^j \sup_T | p(x) | \quad j = 0, 1, 2, \ldots \tag{1}$$

where T is a one dimensional torus and $p(x)$ is a trigonometric polynomial of degree at most n.

To express (1) as a convexity inequality, define E_0 = {Trig. polynomials}, E_1 = {continuous, 2π-periodic functions}, E_θ = {2π-periodic functions p with $D^j p \in E_1$}, and $\theta = \frac{1}{j+1}$. Although the following three expressions are not norms, if we set

$$\| p \|_{E_0} = (\text{degree of } p)^{1/j+1} \quad \| p \|_{E_1} = \sup_T | p(x)|^{1/j+1} \quad \| p \|_{E_\theta} = \sup_T | D^j p(x) |^{1/j+1}$$

We can rewrite the equation (1) above as:

$$\| p \|_{E_\theta} \leq \| p \|_{E_0}^{1-\theta} \| p \|_{E_1}^{\theta}, \quad 0 < \theta \leq 1, p \in E_0 \cap E_1.$$

On the other hand the Jackson Inequality can be expressed as:

$$\inf \sup_T | p(x) - p_0(x) | \leq c n^{-j} \sup_T | D^j p(x) | \tag{2}$$

where inf is taken over all trignomeric polynomials p_0 of degree at most n and p is 2π-periodic, j-times continuously differentiable function.

Using the above notation and setting $p = p_0 + p_1$, we have the following version of (2).

For each $p \in E_\theta$ and for each n, there exists $p_0 \in E_0, p_1, \in E_1$ with $p = p_0 + p_1$ such that

$$\| p_0 \|_{E_0} \leq c n^{\theta} \| p \|_{E_\theta},$$

$$\| p_1 \|_{E_1} \leq c n^{\theta - 1} \| p \|_{E_\theta}.$$

Notice that the above inequalities can also be interpreted as a space possessing K-type θ.

"K-functional" of the real interpolation has its connections with "best approximation" in approximation theory [23]. Given a Banach space E and its linear subspace F, the best approximation to a is $E(a) = E(a; E, F) = \inf_b \| a - b \|_E$. On the other hand if we set $E = E_0 + tE_1$ (direct sum) and $F = \{(b, -b) : b \in E_0 \cap E_1\}$ then

$$K(t; a) = \inf_{b \in F} (\| a_0 - b \|_{E_0} + t \| a_1 + b \|_{E_1}) = E(\vec{a}, E, F)$$

where $\vec{a} = (a_0, a_1)$ has a fixed decomposition as $a = a_0 + a_1$. In the classical theory,

$E_n(a)$ is related to $w(\frac{1}{n}, a)$ (modulus of continuity), therefore it is natural to except results connecting K-functionals to the modulus of continuity, for results of this type we refer to [21], [23].

6.2 s-Numbers and Inequalities of Bernstein-Jackson-Type

There are analogies between s-numbers (Kolmogorov, Gelfand, and approximation number) and Bernstein numbers of functions. Also, entropy numbers (more precisely entropy moduli) of operators and the modulus of continuity of functions are related. In the following we give an example of these analogies [7].

NOTATION. Let $E = L_p^*[0, 1]$, $p \in [1, \infty)$, and $L_\infty^*[0, 1] := C[0, 1]$ denote the space of p-summable and continuous 1-periodic functions respectively. Given $f \in L_p^*[0, 1]$, the n-th Bernstein number $E_n^{(p)}(f)$ is defined as

$$E_n^p(f) := \inf \|f - t\|_p, \quad n = 0, 1, 2, \ldots .$$

where infimum is taken over all trigonometric polynomials t with degree $(t) < n$, $n = 1, 2, \ldots$. Clearly $E_0^{(p)}(f) = \|f\|_p$. If $f \in L_p^*[0, 1]$. The modulus of continuity is defined by

$$w^{(p)}(f, \delta) = \sup_{0<|h|\leq\delta} \left(\int_0^1 |f(z+h) - f(x)|^p \, dx \right)^{1/p}$$

Bernstein Inequalities. The Bernstein Inequality for functions $f \in L_p^*[0, 1]$ says [18]:

$$w^{(p)}(f, \frac{1}{n}) \leq \frac{\delta}{n} \Sigma_1^n E_k^{(p)}(f), \text{ for } n = 1, 2, \ldots .$$

from Theorem 1 we know that for $s \in \{a, c, d\}$ one has analogous inequality between entropy numbers and s-numbers

$$2e_n(T) \leq \frac{\delta}{n} \Sigma_1^n s_k(T), \text{ for } n = 1, 2, \ldots .$$

Jackson Inequalities. The Jackson inequality for functions $f \in L_p^*[0, 1]$ gives [19]:

$$E_n^{(p)}(f) \leq \zeta\, w^{(p)}(f, \frac{1}{n}), \text{ for } n = 1, 2, \ldots .$$

If T acts between Hilbert spaces, one has an analogous inequality for operator, and $s \in \{a, c, d\}$

$$s_n(T) \leq 2\zeta\, e_n(T), \text{ for } n = 1, 2, \dots .$$

EXAMPLE. ([7]) Let $f \in L_p^*[0, 1]$, $p \in [1, \infty]$ define the convolution operator T_f by

$$T_f(g) = f * g = \int_0^1 f(x-y)\, g(y)\, dy$$

This operator T_f may be considered as a map from $L_{p'}^*[0, 1]$ into $C^*[0, 1]$ [31]. Let $s \in \{a, c, d\}$ and $f \in L_p^*[0, 1]$, $p \in [1, \infty)$.Then for $T_f \in \mathcal{L}(L_{p'}^*[0, 1], C^*[0, 1])$ the inequalities

$$s_1(T_f) = \| T_f \| \leq \| f \|_p = E_0^{(p)}(f)$$

$$s_{2n}(T_f) \leq E_n^{(p)}(f), \text{ for } n = 1, 2, \dots$$

$$2e_n(T_f) \leq \zeta^{(r)}(n^{-1/r} \| f \|_p + w^{(p)}(f, \frac{1}{n}))$$

for $0 < r < \infty$, $n = 1, 2, \dots$ are valid.

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